

On bounded solutions of a problem of R. Schilling

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Abstract. It is proved that if

$$0 < q \leq (1 - \sqrt[3]{2} + \sqrt[3]{4})/3,$$

then the zero function is the only solution $f: \mathbb{R} \rightarrow \mathbb{R}$ of (1) satisfying (2) and bounded in a neighbourhood of at least one point of the set (3).

The paper concerns bounded solutions $f: \mathbb{R} \rightarrow \mathbb{R}$ of the functional equation

$$(1) \quad f(qx) = \frac{1}{4q}[f(x-1) + f(x+1) + 2f(x)]$$

such that

$$(2) \quad f(x) = 0 \quad \text{for} \quad |x| > Q$$

where q is a fixed number from the open interval $(0, 1)$ and

$$Q = \frac{q}{1-q}.$$

In what follows any solution $f: \mathbb{R} \rightarrow \mathbb{R}$ of (1) satisfying (2) will be called a *solution of Schilling's problem*. In the present paper we are interested in bounded solutions of Schilling's problem. The first theorem in this direction was obtained by K. Baron in [1]. This theorem reads as follows:

If $q \in (0, \sqrt{2} - 1]$, then the zero function is the only solution of Schilling's problem which is bounded in a neighbourhood of the origin.

This paper generalizes the above theorem in two directions. Namely, the interval $(0, \sqrt{2} - 1]$ is replaced by the larger one

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$(0, \frac{1}{3} - \frac{\sqrt[3]{2}}{3} + \frac{\sqrt[3]{4}}{3}]$ and instead of the boundedness in a neighbourhood of the origin we have boundedness in a neighbourhood of at least one point of the set

$$(3) \quad \left\{ \varepsilon \sum_{i=1}^n q^i \quad : \quad n \in \mathbb{N} \cup \{0, +\infty\}, \varepsilon \in \{-1, 1\} \right\}.$$

(To simplify formulas we adopt the convention $\sum_{i=1}^0 a_i = 0$ for all real sequences $(a_i : i \in \mathbb{N})$.) In other words, we shall prove the following.

Theorem. *If*

$$(4) \quad 0 < q \leq \frac{1}{3} - \frac{\sqrt[3]{2}}{3} + \frac{\sqrt[3]{4}}{3},$$

then the zero function is the only solution of Schilling's problem which is bounded in a neighbourhood of at least one point of the set (3).

The proof of this theorem is based on two lemmas. However, we start with the following simple remarks.

Remark 1. *If f is a solution of Schilling's problem then so is the function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by the formula $g(x) = f(-x)$.*

Remark 2. *Assume that f is a solution of Schilling's problem.*

(i) *If $q \neq \frac{1}{4}$, then $f(Q) = 0$. If $q = \frac{1}{4}$, then $f(Q) = 0$ iff $f(qQ) = 0$.*

(ii) *If $q < \frac{1}{2}$, then $f(0) = 0$.*

Proof. It is enough to put in (1): $x = Q/q$, $x = Q$ and $x = 0$, respectively, and to use condition (2).

Lemma 1. *Assume $q \in (0, \frac{1}{2})$. If a solution of Schilling's problem vanishes either on the interval $(-q, 0)$ or on the interval $(0, q)$, then it vanishes everywhere.*

Proof. Let f be a solution of Schilling's problem vanishing on the interval $[0, q)$. We shall prove that f vanishes on the interval $[0, Q)$. Define a sequence of sets $(A_n : n \in \mathbb{N})$ by the formula:

$$A_n = [0, \sum_{i=1}^n q^i).$$

Fix a positive integer n and suppose that f vanishes on the set A_n . We shall show that f vanishes also on the set A_{n+1} . To this end fix an $x_0 \in A_{n+1} \setminus A_n$. Putting $x = x_0/q$ into (1) and taking into account that $x - 1 \in A_n$, whereas $x + 1 > x > 1 > Q$ we get

$$(5) \quad f(x_0) = \frac{1}{4q}[f(x - 1) + f(x + 1) + 2f(x)] = 0.$$

Consequently f vanishes on the set $\cup_{n=1}^{\infty} A_n$ which equals to $[0, Q)$. This and Remark 2 (i) show that f vanishes on $[0, +\infty)$. Hence and from (1) we infer that f vanishes everywhere.

The case of the interval $(-q, 0)$ reduces to the previous one by Remark 1.

Lemma 2. *Assume $q \in (0, \frac{1}{2})$. If f is a solution of Schilling's problem, then*

$$(6) \quad f(q^{m+n}x + \varepsilon \sum_{i=1}^n q^i) = \left(\frac{1}{2}\right)^n \left(\frac{1}{2q}\right)^{m+n} f(x)$$

for all $x \in (Q - 1, 1 - Q)$, for all $\varepsilon \in \{-1, 1\}$, and for all non-negative integers m and n .

Proof. Fix an $x_0 \in (Q - 1, 1 - Q)$. First we shall show that

$$(7) \quad f(q^m x_0) = \left(\frac{1}{2q}\right)^m f(x_0)$$

for all non-negative integers m . Of course (7) holds for $m = 0$. Suppose that (7) holds for an m . Putting $x = q^m x_0$ into (1) and using (2) and (7) we have

$$\begin{aligned} f(q^{m+1}x_0) &= \frac{1}{4q}[f(x - 1) + f(x + 1) + 2f(x)] = \frac{1}{2q}f(x) \\ &= \left(\frac{1}{2q}\right)^{m+1} f(x_0). \end{aligned}$$

This proves that (7) holds for all non-negative integers m .

Fix now a non-negative integer n and suppose that (6) is satisfied for all $x \in (Q - 1, 1 - Q)$, for all $\varepsilon \in \{-1, 1\}$, and for all non-negative

integers m . Putting $x = q^{m+n}x_0 + \varepsilon \sum_{i=1}^n q^i + \varepsilon$ into (1) and applying (2) and (6) with $x = x_0$ we obtain

$$\begin{aligned} f(q^{m+n+1}x_0 + \varepsilon \sum_{i=1}^{n+1} q^i) &= f(qx) = \frac{1}{4q}[f(x-1) + f(x+1) + 2f(x)] \\ &= \frac{1}{4q}f(x - \varepsilon) = \left(\frac{1}{2}\right)^{n+1} \left(\frac{1}{2q}\right)^{m+n+1} f(x_0). \end{aligned}$$

The proof is completed.

Now we pass to the proof of the main theorem.

Proof of the theorem. It follows from (4) that $q < 1/2$.

Fix $n \in \mathbb{N} \cup \{0, +\infty\}$ and $\varepsilon \in \{-1, 1\}$ such that a solution f of Schilling's problem is bounded in a neighbourhood of $\varepsilon \sum_{i=1}^n q^i$. We may (and we do) assume that n is finite.

If $|x| < 1 - Q$ is fixed, then the left-hand-side of (6) is bounded with respect to m whereas $\lim_{m \rightarrow \infty} (1/2q)^{m+n} = +\infty$. This shows that

$$(8) \quad f(x) = 0 \quad \text{for} \quad |x| < 1 - Q.$$

Consider two cases:

$$(i) \quad q \leq \frac{3-\sqrt{5}}{2}$$

and

$$(ii) \quad \frac{3-\sqrt{5}}{2} < q \leq \frac{1}{3} - \frac{\sqrt[3]{2}}{3} + \frac{\sqrt[3]{4}}{3}.$$

In the case (i) we have $q \leq 1 - Q$ which jointly with (8) and Lemma 1 gives $f = 0$.

So we assume now that (ii) holds. First we notice that putting $x = 1 - Q$ into (1) and applying (8), Remarks 1 and 2 (i) and (2) we get

$$0 = f(q(1-Q)) = \frac{1}{4q}[f(-Q) + f(2-Q) + 2f(1-Q)] = \frac{1}{2q}f(1-Q).$$

Hence, from (8) and Remark 1 we obtain

$$(9) \quad f(x) = 0 \quad \text{for} \quad |x| \leq 1 - Q.$$

Fix an $x_0 \in [qQ, q(2-Q)]$. Putting $x = x_0/q$ into (1) and using (9), (2) and Remark 2 we have (5). Similarly (cf. Remark 1),

$f(x) = 0$ for $x \in [-q(2 - Q), -qQ]$. Consequently,

$$(10) \quad f(x) = 0 \quad \text{whenever} \quad qQ \leq |x| \leq q(2 - Q).$$

Now we fix an $x_0 \in [q - q^2(2 - Q), q^2(2 - Q)]$. Putting $x = x_0/q$ into (1), taking into account the inequality $qQ < 1 - q(2 - Q)$ and applying (10) and (2) we obtain (5) once again. Similarly $f(x) = 0$ for $x \in [-q^2(2 - Q), -q + q^2(2 - Q)]$ and so

$$(11) \quad f(x) = 0 \quad \text{whenever} \quad q - q^2(2 - Q) \leq |x| \leq q^2(2 - Q).$$

As the function $3t^3 - 3t^2 + 3t - 1$ increases and vanishes at $(1 - \sqrt[3]{2} + \sqrt[3]{4})/3$, we have

$$(12) \quad q - q^2(2 - Q) \leq 1 - Q.$$

Relations (9), (12) and (11) give

$$(13) \quad f(x) = 0 \quad \text{for} \quad |x| \leq q^2(2 - Q).$$

Now let us fix an $x_0 \in [1 - q(2 - Q), 1 - qQ]$. Putting $x = x_0 - 1$ into (1) and using (13), (2) and (10) we have

$$(14) \quad 0 = f(qx) = \frac{1}{4q}[f(x - 1) + f(x + 1) + 2f(x)] = \frac{1}{4q}f(x_0).$$

So we obtain

$$(15) \quad f(x) = 0 \quad \text{whenever} \quad 1 - q(2 - Q) \leq x \leq 1 - qQ.$$

Since (cf. (12)) $q + q^2Q < 1 - qQ$ and $1 - q(2 - Q) < q(2 - Q)$, (15) proves that

$$(16) \quad f(x) = 0 \quad \text{whenever} \quad q(2 - Q) \leq x \leq q + q^2Q.$$

Finally assume that $1 - Q \leq x_0 \leq qQ$. Putting $x = x_0 + 1$ into (1) and using (16) and (2) we see that (14) holds. Hence

$$f(x) = 0 \quad \text{whenever} \quad 1 - Q \leq x \leq qQ,$$

which jointly with (9) and (10) gives

$$f(x) = 0 \quad \text{whenever} \quad 0 \leq x \leq q(2 - Q).$$

In particular, since $q < q(2 - Q)$, f vanishes on the interval $(0, q)$. This jointly with Lemma 1 completes the proof.

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Reference

- [1] K. Baron, *On a problem of R.Schilling*. Berichte der Mathematisch-statistischen Sektion in der Forschungsgesellschaft Joanneum–Graz, Bericht Nr. 286 (1988).

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